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# Multiple positive solutions for some $p$-Laplacian boundary value problems ☆

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**Abstract**

This paper deals with the existence of multiple positive solutions for the one-dimensional $p$-Laplacian

$$\left(\varphi_p\left(x'(t)\right)\right)' + q(t)f\left(t, x(t), x'(t)\right) = 0, \quad t \in (0,1),$$

subject to one of the following boundary conditions:

$$\alpha\varphi_p\left(x(0)\right) - \beta\varphi_p\left(x'(0)\right) = 0, \qquad \gamma\varphi_p\left(x(1)\right) + \delta\varphi_p\left(x'(1)\right) = 0,$$

or

$$x(0) - g_1\left(x'(0)\right) = 0, \qquad x(1) + g_2\left(x'(1)\right) = 0,$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. By means of a fixed point theorem due to Avery and Peterson, sufficient conditions are obtained that guarantee the existence of at least three positive solutions. The interesting point is the nonlinear term $f$ is involved with the first-order derivative explicitly.


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## 1. Introduction

In this paper we study the existence of multiple positive solutions for the one-dimensional $p$-Laplacian

$$\big(\varphi_p\big(x'(t)\big)\big)' + q(t) f\big(t, x(t), x'(t)\big) = 0, \quad t \in (0,1), \tag{1.1}$$

subject to one of the following boundary conditions:

$$\alpha\varphi_p\big(x(0)\big) - \beta\varphi_p\big(x'(0)\big) = 0, \qquad \gamma\varphi_p\big(x(1)\big) + \delta\varphi_p\big(x'(1)\big) = 0, \tag{1.2}$$

or

$$x(0) - g_1\big(x'(0)\big) = 0, \qquad x(1) + g_2\big(x'(1)\big) = 0, \tag{1.3}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. $q(t)$ is a nonnegative continuous function defined on $(0,1)$, $f(t,u,v)$ is a nonnegative continuous function defined on $[0,1] \times [0,\infty) \times (-\infty,+\infty)$, $g_1(t)$ and $g_2(t)$ are all continuous functions defined on $(-\infty,+\infty)$; $\alpha > 0$, $\beta \geqslant 0$, $\gamma > 0$, $\delta \geqslant 0$.

Equations of the above form occur in the study of the $n$-dimensional $p$-Laplace equation, non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium [13]. When the nonlinear term $f$ does not depend on the first-order derivative, Eq. (1.1) together with Dirichlet boundary condition $x(0) = x(1) = 0$ and mixed boundary condition $x'(0) = x(1) = 0$ has been studied extensively, and the existence and multiplicity results are available in the literature [1–3,5–12,14,15]. Recently, in [9], Kong and Wang studied the multiplicity for equation

$$\big(\varphi_p\big(x'(t)\big)\big)' + q(t) f\big(t, x(t)\big) = 0, \quad 0 < t < 1, \tag{1.4}$$

subject to nonlinear boundary condition (1.3) by using the theory of fixed point index. In [10], Lü et al. obtained triple positive solutions of Eq. (1.4) subject to Dirichlet boundary condition and mixed boundary condition by using the Leggett–Williams fixed point theorem. In [6,7], He and Ge obtained triple positive solutions for problems (1.4), (1.2) and (1.4), (1.3) by using the Leggett–Williams fixed point theorem and the Krasnosel'skii fixed point theorem, respectively. In [3], by using the five functional fixed point theorem, Avery and Henderson obtained the existence of at least three positive pseudo-symmetric solutions for Eq. (1.4) with three-point boundary condition $u(0) = 0$ and $u(v) = u(1)$, where $v \in (0,1)$. In [15], by using the shooting method, Wong obtain the existence of positive solution for Eq. (1.4) with boundary condition $x'(0) = x(1) = 0$.

However, multiplicity is not available for the case when the nonlinear term is involved in first-order derivative explicitly. This paper fills this gap in the literature. The purpose of this paper is to improve and generalize the results in the above mentioned references. We shall prove that (1.1), (1.2) and (1.1), (1.3) possesses at least three positive solutions.

The following hypotheses are adopted throughout this paper:

(H1) $f \in C([0,1] \times [0,\infty) \times R, [0,\infty))$;

(H2) $q(t)$ is a nonnegative continuous function defined in $(0,1)$, $q(t) \not\equiv 0$ on any subinterval of $(0,1)$. In addition, $\int_0^1 q(t)\,dt < +\infty$.
(H3) $\alpha > 0$, $\beta \geqslant 0$, $\gamma > 0$, $\delta \geqslant 0$.
(H4) $g_1(s)$ and $g_2(s)$ are both nondecreasing continuous odd functions defined on $(-\infty,+\infty)$ and at least one of them satisfies the condition that there exists $B \geqslant 0$ such that

$$g_j(s) = Bs \quad \text{for all } s \geqslant 0,\ \ j = 1 \text{ or } 2.$$

Our main results will depend on an application of a fixed point theorem due to Avery and Peterson which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis here is that the nonlinear term is involved explicitly with the first-order derivative.

## 2. Background material and definitions

For the convenience of the reader, we provide some background material from the theory of cones in Banach spaces. We also state in this section the Avery–Peterson fixed point theorem.

**Definition 2.1.** Let $E$ be a real Banach space over $R$. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geqslant 0$, $b \geqslant 0$, and
(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by

$$x \leqslant y \quad \text{if and only if} \quad y - x \in P.$$

**Definition 2.2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Definition 2.3.** The map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha\bigl(tx + (1-t)y\bigr) \geqslant t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$. Similarly, we say the map $\gamma$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ provided that $\gamma : P \to [0,\infty)$ is continuous and

$$\gamma\bigl(tx + (1-t)y\bigr) \leqslant t\gamma(x) + (1-t)\gamma(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$.

Let $\gamma$ and $\theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$. Then for positive real numbers $a$, $b$, $c$, and $d$, we define the following convex sets:

$$P(\gamma, d) = \{x \in P \mid \gamma(x) < d\},$$
$$P(\gamma, \alpha, b, d) = \{x \in P \mid b \leqslant \alpha(x),\ \gamma(x) \leqslant d\},$$
$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P \mid b \leqslant \alpha(x),\ \theta(x) \leqslant c,\ \gamma(x) \leqslant d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P \mid a \leqslant \psi(x),\ \gamma(x) \leqslant d\}.$$

The following fixed point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

**Theorem 2.1** [4]. *Let $P$ be a cone in a real Banach space $E$. Let $\gamma$ and $\theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$ satisfying $\psi(\lambda x) \leqslant \lambda\psi(x)$ for $0 \leqslant \lambda \leqslant 1$, such that for some positive numbers $M$ and $d$,*

$$\alpha(x) \leqslant \psi(x) \quad \text{and} \quad \|x\| \leqslant M\gamma(x) \tag{2.1}$$

*for all $x \in \overline{P(\gamma, d)}$. Suppose*

$$T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$$

*is completely continuous and there exist positive numbers $a$, $b$, and $c$ with $a < b$ such that*

(S1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$;
(S2) $\alpha(Tx) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$;
(S3) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$.

*Then $T$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$, such that*

$$\gamma(x_i) \leqslant d \quad \text{for } i = 1, 2, 3,$$
$$b < \alpha(x_1),$$
$$a < \psi(x_2), \quad \text{with } \alpha(x_2) < b,$$

*and*

$$\psi(x_3) < a.$$

## 3. Existence of triple positive solutions to (1.1), (1.2)

In this section, we impose growth conditions on $f$ which allow us to apply Theorem 2.1 to establish the existence of triple positive solutions of problem (1.1), (1.2). It follows from (H2) that there exists a natural number $k \geqslant 3$ such that $0 < \int_{1/k}^{(k-1)/k} q(t)\,dt < +\infty$.

Let $X = C^1[0,1]$ be endowed with the ordering $x \leqslant y$ if $x(t) \leqslant y(t)$ for all $t \in [0,1]$, and the maximum norm,

$$\|x\| = \max\Big\{\max_{0\leqslant t\leqslant 1}\big|x(t)\big|, \max_{0\leqslant t\leqslant 1}\big|x'(t)\big|\Big\}.$$

From the fact $(\varphi_p(x'(t)))' = -q(t)f(t,x(t),x'(t)) \leqslant 0$, we know that $x$ is concave on $[0,1]$. So, define the cone $P_1 \subset X$ by

$$P_1 = \big\{x \in X \mid x(t) \geqslant 0,\ \alpha\varphi_p\big(x(0)\big) - \beta\varphi_p\big(x'(0)\big) = 0,\\ \gamma\varphi_p\big(x(1)\big) + \delta\varphi_p\big(x'(1)\big) = 0,\ x \text{ is concave on } [0,1]\big\} \subset X.$$

Let the nonnegative continuous concave functional $\alpha_1$, the nonnegative continuous convex functional $\theta_1, \gamma_1$, and the nonnegative continuous functional $\psi_1$ be defined on the cone $P_1$ by

$$\gamma_1(x) = \max_{0\leqslant t\leqslant 1}\big|x'(t)\big|, \qquad \psi_1(x) = \theta_1(x) = \max_{0\leqslant t\leqslant 1}\big|x(t)\big|,$$
$$\alpha_1(x) = \min_{1/k\leqslant t\leqslant (k-1)/k}\big|x(t)\big| \quad \text{for } x \in P_1.$$

In our main results, we will make use of the following lemma.

**Lemma 3.1.** *For $x \in P_1$, there exists a constant $M > 0$ such that*

$$\max_{0\leqslant t\leqslant 1}\big|x(t)\big| \leqslant M \max_{0\leqslant t\leqslant 1}\big|x'(t)\big|.$$

**Proof.** By the concavity of $x$, there is

$$x(t) - x(0) \leqslant x'(0) \leqslant \max_{0\leqslant t\leqslant 1}\big|x'(t)\big|, \quad t \in [0,1], \tag{3.1}$$
$$x(t) - x(1) \leqslant -x'(1) \leqslant \max_{0\leqslant t\leqslant 1}\big|x'(t)\big|, \quad t \in [0,1]. \tag{3.2}$$

On the other hand, as $\alpha > 0$, taking into account that $x$ is nonnegative and assumption (H3), we have

$$x(0) = \varphi_p^{-1}\left(\frac{\beta}{\alpha}\right)x'(0) \leqslant \varphi_p^{-1}\left(\frac{\beta}{\alpha}\right)\max_{0\leqslant t\leqslant 1}\big|x'(t)\big|; \tag{3.3}$$

as $\gamma > 0$, there is

$$x(1) = \varphi_p^{-1}\left(-\frac{\delta}{\gamma}\right)x'(1) \leqslant \varphi_p^{-1}\left(\frac{\delta}{\gamma}\right)\max_{0\leqslant t\leqslant 1}\big|x'(t)\big|. \tag{3.4}$$

Thus, we have

$$\max_{0\leqslant t\leqslant 1}\big|x(t)\big| \leqslant \left(1 + \varphi_p^{-1}\left(\frac{\beta}{\alpha}\right)\right)\max_{0\leqslant t\leqslant 1}\big|x'(t)\big|,$$
$$\max_{0\leqslant t\leqslant 1}\big|x(t)\big| \leqslant \left(1 + \varphi_p^{-1}\left(\frac{\delta}{\gamma}\right)\right)\max_{0\leqslant t\leqslant 1}\big|x'(t)\big|.$$

Therefore, setting

$$M=\min\left\{1+\varphi_p^{-1}\left(\frac{\beta}{\alpha}\right),1+\varphi_p^{-1}\left(\frac{\delta}{\gamma}\right)\right\},$$

the proof is complete. □

With Lemma 3.1 and the concavity of $x$, for all $x\in P_1$, the functionals defined above hold relations

$$\frac{1}{k}\theta_1(x)\leqslant\alpha_1(x)\leqslant\theta_1(x)=\psi_1(x),\qquad \|x\|=\max\{\theta_1(x),\gamma_1(x)\}\leqslant M\gamma_1(x). \quad (3.5)$$

Therefore, the condition (2.1) of Theorem 2.1 is satisfied.

For $x\in P_1$, there is $x'(0)\geqslant 0$ and $x'(1)\leqslant 0$, therefore there exists a constant $\sigma$ $(=\sigma_x)$ such that $x'(\sigma)=0$. Define an operator $T:P_1\to P_1$ by

$$(Tx)(t):=\begin{cases}\varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_0^\sigma q(r)f(r,x(r),x'(r))\,dr\right)\\ \quad+\int_0^t\varphi_p^{-1}\left(\int_s^\sigma q(r)f(r,x(r),x'(r))\,dr\right)ds, & \text{for } 0\leqslant t\leqslant\sigma,\\ \varphi_p^{-1}\left(\frac{\delta}{\gamma}\int_\sigma^1 q(r)f(r,x(r),x'(r))\,dr\right)\\ \quad+\int_t^1\varphi_p^{-1}\left(\int_\sigma^s q(r)f(r,x(r),x'(r))\,dr\right)ds, & \text{for } \sigma\leqslant t\leqslant 1.\end{cases} \quad (3.6)$$

From the definition of $T$, we deduce that for each $x\in P_1$, there is $Tx\in C^1[0,1]$ is nonnegative and satisfies (1.2). Moreover, $(Tx)(\sigma)$ is the maximum value of $Tx$ on $[0,1]$, since

$$(Tx)'(t)=\begin{cases}\varphi_p^{-1}\left(\int_t^\sigma q(r)f(r,x(r),x'(r))\,dr\right), & \text{for } 0\leqslant t\leqslant\sigma,\\ -\varphi_p^{-1}\left(\int_\sigma^t q(r)f(r,x(r),x'(r))\,dr\right), & \text{for } \sigma\leqslant t\leqslant 1,\end{cases} \quad (3.7)$$

is continuous and nonincreasing in $[0,1]$ and $(Tx)'(\sigma)=0$. As $(Tx)'$ is nonincreasing on $[0,1]$, we have $Tx\in P_1$. This show that $T(P_1)\subset P_1$, and that each fixed point of $T$ is a solution of problem (1.1), (1.2). By similar arguments in [10,15], $T:P_1\to P_1$ is completely continuous.

Let

$$C_1=\varphi_p^{-1}\left(\int_0^1 q(r)\,dr\right),$$

$$M_1=\max\left\{\int_0^{1/2}\varphi_p^{-1}\left(\int_s^{1/2}q(r)\,dr\right)ds+\varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_0^{1/2}q(r)\,dr\right),\right.$$
$$\left.\int_{1/2}^1\varphi_p^{-1}\left(\int_{1/2}^s q(r)\,dr\right)ds+\varphi_p^{-1}\left(\frac{\delta}{\gamma}\int_{1/2}^1 q(r)\,dr\right)\right\},$$

$$N_1 = \min\left\{ \int_{1/k}^{1/2} \varphi_p^{-1}\left(\int_s^{1/2} q(r)\,dr\right) ds + \varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_{1/k}^{1/2} q(r)\,dr\right), \int_{1/2}^{(k-1)/k} \varphi_p^{-1}\left(\int_{1/2}^{s} q(r)\,dr\right) ds + \varphi_p^{-1}\left(\frac{\delta}{\gamma}\int_{1/2}^{(k-1)/k} q(r)\,dr\right)\right\}.$$

We are now ready to apply the Avery–Peterson fixed point theorem to the operator $T$ to give sufficient conditions for the existence of at least three positive solutions to problem (1.1), (1.2).

**Theorem 3.1.** *Assume* (H1), (H2), *and* (H3) *hold. Let* $0 < a < b \leqslant Md/k$, *and suppose that* $f$ *satisfies the following conditions*:

(A1) $f(t,u,v) \leqslant \varphi_p(d/C_1)$, *for* $(t,u,v) \in [0,1] \times [0,Md] \times [-d,d]$;
(A2) $f(t,u,v) > \varphi_p(kb/N_1)$, *for* $(t,u,v) \in [1/k,(k-1)/k] \times [b,kb] \times [-d,d]$;
(A3) $f(t,u,v) < \varphi_p(a/M_1)$, *for* $(t,u,v) \in [0,1] \times [0,a] \times [-d,d]$.

*Then the boundary value problem* (1.1), (1.2) *has at least three positive solutions* $x_1$, $x_2$, *and* $x_3$ *such that*

$$\max_{0\leqslant t\leqslant 1}\left|x_i'(t)\right| \leqslant d \quad \textit{for } i = 1,2,3,$$

$$b < \min_{1/k\leqslant t\leqslant (k-1)k}\left|x_1(t)\right|, \qquad \max_{0\leqslant t\leqslant 1}\left|x_1(t)\right| \leqslant Md,$$

$$a < \max_{0\leqslant t\leqslant 1}\left|x_2(t)\right| < kb, \quad \textit{with} \min_{1/k\leqslant t\leqslant (k-1)/k}\left|x_2(t)\right| < b,$$

*and*

$$\max_{0\leqslant t\leqslant 1}\left|x_3(t)\right| < a.$$

**Proof.** Problem (1.1), (1.2) has a solution $x = x(t)$ if and only if $x$ solves the operator equation $x = Tx$. Thus we set out to verify that the operate $T$ satisfies the Avery–Peterson fixed point theorem which will prove the existence of three fixed points of $T$ which satisfy the conclusion of the theorem.

For $x \in \overline{P_1(\gamma_1, d)}$, there is $\gamma_1(x) = \max_{0\leqslant t\leqslant 1}|x'(t)| \leqslant d$. With Lemma 3.1, there is $\max_{0\leqslant t\leqslant 1}|x(t)| \leqslant Md$, then condition (A1) implies $f(t,x(t),x'(t)) \leqslant \varphi_p(d/C_1)$. On the other hand, for $x \in P_1$, there is $Tx \in P_1$, then $Tx$ is concave on $[0,1]$, and $\max_{t\in[0,1]}|(Tx)'(t)| = \max\{|(Tx)'(0)|, |(Tx)'(1)|\}$, so

$$\gamma_1(Tx) = \max_{t\in[0,1]}\left|(Tx)'(t)\right|$$

$$= \max\left\{\varphi_p^{-1}\left(\int_0^{\sigma} q(r) f\bigl(r, x(r), x'(r)\bigr)\, dr\right),\right.$$

$$\left.\varphi_p^{-1}\left(\int_{\sigma}^{1} q(r) f\bigl(r, x(r), x'(r)\bigr)\, dr\right)\right\}$$

$$\leqslant \frac{d}{C_1}\varphi_p^{-1}\left(\int_0^1 q(r)\, dr\right) = \frac{d}{C_1} C_1 = d.$$

Therefore, $T : \overline{P_1(\gamma_1, d)} \to \overline{P_1(\gamma_1, d)}$.

To check condition (S1) of Theorem 2.1, we choose $x(t) = kb/2, 0 \leqslant t \leqslant 1$. It is easy to see that $x(t) = kb/2 \in P_1(\gamma_1, \theta_1, \alpha_1, b, kb, d)$ and $\alpha_1(x) = \alpha_1(kb/2) > b$, and so $\{x \in P_1(\gamma_1, \theta_1, \alpha_1, b, kb, d) \mid \alpha_1(x) > b\} \neq \emptyset$. Hence, for $x \in P_1(\gamma_1, \theta_1, \alpha_1, b, kb, d)$, there is $b \leqslant x(t) \leqslant kb$, $|x'(t)| \leqslant d$ for $1/k \leqslant t \leqslant (k-1)/k$. Thus, by condition (A2) of this theorem, we have $f(t, x(t), x'(t)) \geqslant \varphi_p(kb/N_1)$ for $1/k \leqslant t \leqslant (k-1)/k$, and combining the conditions of $\alpha_1$ and $P_1$, we have

$$\alpha_1(Tx) = \min_{1/k \leqslant t \leqslant (k-1)/k} \bigl|(Tx)(t)\bigr| \geqslant \frac{1}{k}\|Tx\|$$

$$= \frac{1}{k}\int_0^{\sigma} \varphi_p^{-1}\left(\int_s^{\sigma} q(r) f\bigl(r, x(r), x'(r)\bigr)\, dr\right) ds$$

$$+ \frac{1}{k}\varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_0^{\sigma} q(r) f\bigl(r, x(r), x'(r)\bigr)\, dr\right)$$

$$= \frac{1}{k}\int_{\sigma}^{1} \varphi_p^{-1}\left(\int_{\sigma}^{s} q(r) f\bigl(r, x(r), x'(r)\bigr)\, dr\right) ds$$

$$+ \frac{1}{k}\varphi_p^{-1}\left(\frac{\delta}{\gamma}\int_{\sigma}^{1} q(r) f\bigl(r, x(r), x'(r)\bigr)\, dr\right)$$

$$\geqslant \frac{1}{k}\min\left\{\int_0^{1/2} \varphi_p^{-1}\left(\int_s^{1/2} q(r) f\bigl(r, x(r), x'(r)\bigr)\, dr\right) ds\right.$$

$$+ \varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_0^{1/2} q(r) f\bigl(r, x(r), x'(r)\bigr)\, dr\right),$$

$$\int_{1/2}^{1} \varphi_p^{-1}\left(\int_{1/2}^{s} q(r) f\bigl(r, x(r), x'(r)\bigr)\, dr\right) ds$$

$$+\varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_{1/2}^{1} q(r)f\big(r,x(r),x'(r)\big)\,dr\right)\Bigg\}$$

$$\geqslant \frac{1}{k}\min\Bigg\{\int_{1/k}^{1/2}\varphi_p^{-1}\left(\int_s^{1/2} q(r)f\big(r,x(r),x'(r)\big)\,dr\right)ds$$

$$+\varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_{1/k}^{1/2} q(r)f\big(r,x(r),x'(r)\big)\,dr\right),$$

$$\int_{1/2}^{(k-1)/k}\varphi_p^{-1}\left(\int_{1/2}^{s} q(r)f\big(r,x(r),x'(r)\big)\,dr\right)ds$$

$$+\varphi_p^{-1}\left(\frac{\delta}{\gamma}\int_{1/2}^{(k-1)/k} q(r)f\big(r,x(r),x'(r)\big)\,dr\right)\Bigg\}$$

$$> \frac{1}{k}\frac{kb}{N_1}N_1 = b,$$

i.e.,

$$\alpha_1(Tx) > b \quad \text{for all } x\in P_1(\gamma_1,\theta_1,\alpha_1,b,kb,d).$$

This show that condition (S1) of Theorem 2.1 is satisfied.

Secondly, with (3.5), we have

$$\alpha_1(Tx)\geqslant \frac{1}{k}\theta_1(Tx) > \frac{1}{k}kb = b \quad \text{for all } x\in P_1(\gamma_1,\alpha_1,b,d) \text{ with } \theta_1(Tx) > kb.$$

Thus, condition (S2) of Theorem 2.1 is satisfied.

Finally, we show that condition (S3) of Theorem 2.1 also holds. Clearly, as $\psi_1(0) = 0 < a$, there holds $0 \notin R(\gamma_1,\psi_1,a,d)$. Suppose that $x\in R(\gamma_1,\psi_1,a,d)$ with $\psi_1(x) = a$. Then, by the condition (A3) of this theorem,

$$\psi_1(Tx) = \max_{0\leqslant t\leqslant 1}\big|(Tx)(t)\big|$$

$$= \int_0^{\sigma}\varphi_p^{-1}\left(\int_s^{\sigma} q(r)f\big(r,x(r),x'(r)\big)\,dr\right)ds$$

$$+\varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_0^{\sigma} q(r)f\big(r,x(r),x'(r)\big)\,dr\right)$$

$$= \int_{\sigma}^{1}\varphi_p^{-1}\left(\int_{\sigma}^{s} q(r)f\big(r,x(r),x'(r)\big)\,dr\right)ds$$

$$
\begin{aligned}
&+\varphi_p^{-1}\left(\frac{\delta}{\gamma}\int_0^{\sigma} q(r) f\big(r, x(r), x'(r)\big)\,dr\right)\\
&\leqslant \max\left\{\int_0^{1/2}\varphi_p^{-1}\left(\int_s^{1/2} q(r) f\big(r, x(r), x'(r)\big)\,dr\right)ds\right.\\
&\qquad +\varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_0^{1/2} q(r) f\big(r, x(r), x'(r)\big)\,dr\right),\\
&\qquad \int_{\frac{1}{2}}^{1}\varphi_p^{-1}\left(\int_{1/2}^{s} q(r) f\big(r, x(r), x'(r)\big)\,dr\right)ds\\
&\qquad \left. +\varphi_p^{-1}\left(\frac{\delta}{\gamma}\int_{1/2}^{1} q(r) f\big(r, x(r), x'(r)\big)\,dr\right)\right\}\\
&< \frac{a}{M_1}\max\left\{\int_0^{1/2}\varphi_p^{-1}\left(\int_s^{1/2} q(r)\,dr\right)ds+\varphi_p^{-1}\left(\frac{\beta}{\alpha}\int_0^{1/2} q(r)\,dr\right),\right.\\
&\qquad \left.\int_{1/2}^{1}\varphi_p^{-1}\left(\int_{1/2}^{s} q(r)\,dr\right)ds+\varphi_p^{-1}\left(\frac{\delta}{\gamma}\int_{1/2}^{1} q(r)\,dr\right)\right\}\\
&= a.
\end{aligned}
$$

So, the condition (S3) of Theorem 2.1 is satisfied. On the other hand, for $x \in P_1$, (3.5) holds. Therefore, an application of Theorem 2.1 implies the boundary value problem (1.1), (1.2) has at least three positive solutions $x_1$, $x_2$, and $x_3$ such that

$$
\begin{aligned}
&\max_{0\leqslant t\leqslant 1}\left|x_i'(t)\right| \leqslant d \quad \text{for } i = 1, 2, 3,\\
&b < \min_{1/k\leqslant t\leqslant (k-1)/k}\left|x_1(t)\right|, \qquad \max_{0\leqslant t\leqslant 1}\left|x_1(t)\right| \leqslant Md,\\
&a < \max_{0\leqslant t\leqslant 1}\left|x_2(t)\right| < kb, \quad \text{with} \min_{1/k\leqslant t\leqslant (k-1)/k}\left|x_2(t)\right| < b,
\end{aligned}
$$

and

$$
\max_{0\leqslant t\leqslant 1}\left|x_3(t)\right| < a.
$$

The proof is complete. □

**Example 3.1.** Consider the boundary value problem

$$
\big(|x'|x'\big)' + f\big(t, x(t), x'(t)\big) = 0, \quad 0 < t < 1, \tag{3.8}
$$

$$
x(0) = x(1) = 0, \tag{3.9}
$$

where

$$f(t,u,v)=\begin{cases}\sin t+2306u^{10}+\frac{1}{6}\left(\frac{v}{50000}\right)^3, & \text{for } u\leqslant 4,\\ \sin t+2306\cdot 4^{10}+\frac{1}{6}\left(\frac{v}{50000}\right)^3, & \text{for } u>4.\end{cases}$$

Choose $a=\frac{1}{2}$, $b=1$, $k=4$, $d=50000$; we note $C_1=1$, $M_1=\sqrt{2}/6$, $N_1=1/12$. Consequently, $f(t,u,v)$ satisfy

$$f(t,u,v)<\varphi_z\left(\frac{a}{M_1}\right)=4.5 \quad \text{for } 0\leqslant t\leqslant 1,\ 0\leqslant u\leqslant 1/2,\ -50000\leqslant v\leqslant 50000,$$

$$f(t,u,v)>\varphi_z\left(\frac{4b}{N_1}\right)=2304 \quad \text{for } 1/4\leqslant t\leqslant 3/4,\ 1\leqslant u\leqslant 4,\ -50000\leqslant v\leqslant 50000,$$

$$f(t,u,v)<\varphi_z\left(\frac{d}{C_1}\right)=2.5\times 10^9 \quad \text{for } 0\leqslant t\leqslant 1,\ 0\leqslant u\leqslant 11786,\ -50000\leqslant v\leqslant 50000.$$

Then all conditions of Theorem 3.1 hold. Thus, with Theorem 3.1, problem (3.8), (3.9) has at least three positive solutions $x_1, x_2, x_3$ such that

$$\max_{0\leqslant t\leqslant 1}\left|x_i'(t)\right|\leqslant 50000 \quad \text{for } i=1,2,3,$$

$$1<\min_{1/4\leqslant t\leqslant 3/4}\left|x_1(t)\right|, \qquad \max_{0\leqslant t\leqslant 1}\left|x_1(t)\right|\leqslant 11786,$$

$$\frac{1}{2}<\max_{0\leqslant t\leqslant 1}\left|x_2(t)\right|<4, \quad \text{with} \quad \min_{1/4\leqslant t\leqslant 3/4}\left|x_2(t)\right|<1,$$

and

$$\max_{0\leqslant t\leqslant 1}\left|x_3(t)\right|<\frac{1}{2}.$$

**Remark 3.1.** The early results, see [2,6–10], for example, are not applicable to the above problem. In conclusion, we see that the nonlinear term is involved in first-order derivative explicitly. In addition, by adopted the ideas used in [10], the constants $M_1$ and $N_1$ chosen here are better than which chosen in [7].

## 4. Existence of triple positive solutions to (1.1), (1.3)

Now we deal with problem (1.1), (1.3). The method is just similar to what we have done in Section 3, so we omit the proof of main result of this section. Without loss of generality, suppose $g_1(s)=Bs$ for some $B\geqslant 0$ and define the cone $P_2\subset X$ by

$$P_2=\left\{x\in X \mid x(t)\geqslant 0,\ x(0)-g_1\left(x'(0)\right)=0,\ x \text{ is concave on } [0,1]\right\}.$$

Let the nonnegative continuous concave functional $\alpha_2$, the nonnegative continuous convex functional $\theta_2, \gamma_2$, and the nonnegative continuous functional $\psi_2$ be defined on the cone $P_2$ by

$$\gamma_2(x) = \max_{t\in[0,1]} \left|x'(t)\right|, \qquad \psi_2(x) = \theta_2(x) = \max_{t\in[0,1]} \left|x(t)\right|,$$
$$\alpha_2(x) = \min_{t\in[1/k,(k-1)/k]} \left|x(t)\right| \quad \text{for } x \in P_2.$$

**Lemma 4.1.** *Assume there exist $B \geqslant 0$ such that $g_i(s) = Bs$, $i = 1$ or $2$, for $s \geqslant 0$. Then, for $x \in P_2$, there holds*

$$\max_{0\leqslant t\leqslant 1} \left|x(t)\right| \leqslant (1+B) \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|.$$

**Proof.** By the concavity of $x$, there is

$$x(t) - x(0) \leqslant x'(0) \leqslant \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|, \quad t \in [0,1], \tag{4.1}$$
$$x(t) - x(1) \leqslant -x'(1) \leqslant \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|, \quad t \in [0,1]. \tag{4.2}$$

On the other hand, if $g_1$ satisfies $g_1(s) = Bs$, taking into account that $x$ is nonnegative, we have

$$x(0) = g_1\big(x'(0)\big) = Bx'(0) \leqslant B \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|; \tag{4.3}$$

if $g_2$ satisfies $g_2(s) = Bs$, taking into account that $x$ is nonnegative, we have

$$x(1) = -g_2\big(x'(1)\big) = g_2\big(-x'(1)\big) = -Bx'(1) \leqslant B \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|. \tag{4.4}$$

Thus, there is

$$\max_{0\leqslant t\leqslant 1} \left|x(t)\right| \leqslant (1+B) \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|, \quad \text{if } g_1(s) = Bs,$$
$$\max_{0\leqslant t\leqslant 1} \left|x(t)\right| \leqslant (1+B) \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|, \quad \text{if } g_2(s) = Bs.$$

Therefore, the proof is complete. □

With Lemma 4.1 and the concavity of $x$, the functionals defined above hold relations

$$\frac{1}{k}\theta_2(x) \leqslant \alpha_2(x) \leqslant \theta_2(x) = \psi_2(x),$$
$$\|x\| = \max\big\{\theta_2(x), \gamma_2(x)\big\} \leqslant (1+B)\gamma_2(x) \tag{4.5}$$

for all $x \in P_2$.

Define operator $T : P_2 \to P_2$ by

$$(Tx)(t) := \begin{cases} g_1 \circ \varphi_p^{-1}\big(\int_0^\sigma q(r) f(r, x(r), x'(r))\, dr\big) \\ \quad + \int_0^t \varphi_p^{-1}\big(\int_s^\sigma q(r) f(r, x(r), x'(r))\, dr\big)\, ds, & \text{for } 0 \leqslant t \leqslant \sigma, \\ g_2 \circ \varphi_p^{-1}\big(\int_\sigma^1 q(r) f(r, x(r), x'(r))\, dr\big) \\ \quad + \int_t^1 \varphi_p^{-1}\big(\int_\sigma^s q(r) f(r, x(r), x'(r))\, dr\big)\, ds, & \text{for } \sigma \leqslant t \leqslant 1, \end{cases} \tag{4.6}$$

where $\sigma = 0$ if $(Tx)'(0) = 0$, $\sigma = 1$ if $(Tx)'(1) = 0$; otherwise, $\sigma$ is a solution of the equation

$$z_1(x) - z_2(x) = 0; \tag{4.7}$$

here

$$z_1(x) = g_1 \circ \varphi_p^{-1}\left(\int_0^x q(r) f\big(r, x(r), x'(r)\big)\, dr\right) + \int_0^x \varphi_p^{-1}\left(\int_s^x q(r) f\big(r, x(r), x'(r)\big)\, dr\right) ds,$$

$$z_2(x) = g_2 \circ \varphi_p^{-1}\left(\int_x^1 q(r) f\big(r, x(r), x'(r)\big)\, dr\right) + \int_x^1 \varphi_p^{-1}\left(\int_x^s q(r) f\big(r, x(r), x'(r)\big)\, dr\right) ds.$$

As shown in [9], $\sigma$ exists and the operator $T$ is well defined with $\max_{0\leqslant t\leqslant 1} |(Tx)(t)| = (Tx)(\sigma)$. Moreover, a standard argument shows that $T : P_2 \to P_2$ is completely continuous, and each fixed point of $T$ in $P_2$ is a solution of problem (1.1), (1.3).

Let

$$C_2 = \varphi_p^{-1}\left(\int_0^1 q(r)\, dr\right),$$

$$M_2 = \int_0^1 \varphi_p^{-1}\left(\int_s^1 q(r)\, dr\right) ds + B\varphi_p^{-1}\left(\int_0^1 q(r)\, dr\right),$$

$$N_2 = \min\left\{\int_{1/k}^{1/2} \varphi_p^{-1}\left(\int_s^{1/2} q(r)\, dr\right) ds,\ \int_{1/2}^{(k-1)/k} \varphi_p^{-1}\left(\int_{1/2}^{s} q(r)\, dr\right) ds\right\}.$$

**Theorem 4.1.** *Assume* (H1), (H2), *and* (H4) *hold. Let* $0 < a < b \leqslant (1 + B)d/k$, *and suppose that* $f$ *satisfies the following conditions*:

(B1) $f(t, u, v) \leqslant \varphi_p(d/C_2)$, *for* $(t, u, v) \in [0, 1] \times [0, (1 + B)d] \times [-d, d]$;
(B2) $f(t, u, v) > \varphi_p(kb/N_2)$, *for* $(t, u, v) \in [1/k, (k - 1)/k] \times [b, kb] \times [-d, d]$;
(B3) $f(t, u, v) < \varphi_p(a/M_2)$, *for* $(t, u, v) \in [0, 1] \times [0, a] \times [-d, d]$.

*Then the boundary value problem* (1.1), (1.3) *has at least three positive solutions* $x_1$, $x_2$, *and* $x_3$ *such that*

$$\max_{0\leqslant t\leqslant 1}\left|x_i'(t)\right| \leqslant d \quad \text{for } i = 1, 2, 3,$$
$$b < \min_{1/k\leqslant t\leqslant (k-1)/k}\left|x_1(t)\right|, \qquad \max_{0\leqslant t\leqslant 1}\left|x_1(t)\right| \leqslant (1+B)d,$$
$$a < \max_{0\leqslant t\leqslant 1}\left|x_2(t)\right| < kb, \quad \text{with} \min_{1/k\leqslant t\leqslant (k-1)k}\left|x_2(t)\right| < b,$$

*and*

$$\max_{0\leqslant t\leqslant 1}\left|x_3(t)\right| < a.$$

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